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Numerical Solution of One-Dimensional Differential Equations by Weighted Residual Method Using Bernoulli Wavelets

Lingaraj Angadi*

Department of Mathematics, Shri Siddeshwar Government First Grade College & P. G. Studies Centre, Nargund – 582207, India; angadi.lm@gmail.com.

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Abstract

This paper presents the method of solving one-dimensional differential equations through the weighted residual technique, employing Bernoulli wavelets as the basis functions. These wavelets serve as the foundation for the calculation of numerical solutions for one-dimensional differential equations. The numerical outcomes are contrasted with those from current techniques and the precise solution. A selection of numerical test problems is included to demonstrate the practicality and efficiency of the proposed approach.

Keywords: Weighted residual method, Bernoulli wavelets, One-dimensional differential equations, Boundary conditions, Finite difference method.

1 | Introduction

In the fields of science and engineering, numerous issues arise, leading to the formulation of both linear and non-linear second-order differential equations, each with distinct boundary conditions. These equations are tackled through either analytical or numerical approaches. The use of numerical simulations in engineering science and applied mathematics has emerged as a formidable method for representing physical phenomena, especially when analytical solutions prove elusive. The literature on solving these kinds of problems showcases a variety of strategies aimed at achieving higher precision quickly. However, finding analytical solutions to these equations is a rarity. The available literature covers a range of methods for the numerical resolution of these equations [1–3].

Corresponding Author: angadi.lm@gmail.com

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A wavelet, as its name implies, is a miniature wave. Numerous statistical events exhibit a wavelet pattern. Frequently, there are brief episodes of high-frequency wavelets interspersed with lower-frequency ones or the reverse. The concept of wavelet reconstruction aids in pinpointing and recognizing these clusters of small waves, contributing to a deeper comprehension of the causes behind these events. Wavelet theory diverges from Fourier analysis and spectral theory because it relies on a frequency representation that is localized.

One of the most well-known techniques for determining numerical solutions to ordinary and partial differential equations is the Galerkin method. It is ideal for numerous applications due to its simplicity. By employing a compactly supported orthogonal functional basis, the wavelet-Galerkin approach improves upon the conventional Galerkin method. In applied mathematics, Galerkin's approach with wavelet basis is highly acclaimed for its efficiency and usefulness as well as its notable benefits over the conventional finite difference and finite element methods [4], [5].

In this work, the solution is represented by Bernoulli wavelets with undetermined coefficients. The differential equation can be numerically solved by Galerkin's approach in combination with the features of Bernoulli wavelets to identify the unknown coefficients.

The following is an outline of the paper's structure. An overview of Bernoulli wavelets and function approximation is given in Section 2. The Weighted Residual Approach Using Bernoulli Wavelets (WRMBW) is the subject of Section 3. There is a numerical example in Section 4. Finally, a review of the findings from the research is presented in Section 5.

2 | Bernoulli Wavelets and Function Approximation

Bernoulli wavelets $\psi_{n,m}(x) = \psi(k, \hat{n}, m, x)$ have four arguments: $\hat{n} = n-1$, $n = 1, 2, ..., 2^{k-1}$, k is assumed to be any positive integer, m is the order for Bernoulli polynomials and x is the normalized time [6], [7]. They are defined on the interval $[0, 1)$ as follows:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k-1}{2}} \tilde{B}_m(2^{k-1}x - n + 1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

with

$$\tilde{B}_m(x) = \begin{cases} 1, & m = 0, \\ \dfrac{1}{\sqrt{\left(\dfrac{(-1)^{m-1}(m!)^2}{(2m)!}\right)\alpha_{2m}}}, & m > 0, \end{cases} \tag{2.2}$$

where $m = 0,1,2,...,M-1$ and $n = 1,2,...,2^{k-1}$. The coefficient $\dfrac{1}{\sqrt{\left(\dfrac{(-1)^{m-1}(m!)^2}{(2m)!}\right)\alpha_{2m}}}$ is for normality, the dilation parameter is $a = 2^{-(k-1)}$ and the translation parameter is $b = (n-1)2^{-(k-1)}$.

For instance, for $k = 1$ and $M = 3$, we get the Bernoulli wavelet bases as follows:

I. $\psi_{1,0}(x) = 1$,

II. $\psi_{1,1}(x) = \sqrt{3}(2x - 1)$,

III. $\psi_{1,2}(x) = \sqrt{5}(6x^2 - 6x + 1)$, and so on.

Function approximation

Suppose $u(x) \in L^2[0\ ,\ 1)$ is expanded in terms of Bernoulli wavelets as

$$u(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x). \tag{2.3}$$

Truncating the above infinite series, we get

$$u(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x). \tag{2.4}$$

3 | Method of Solution

The one-dimensional differential equation is of the form,

$$\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x} + \beta u = f(x). \tag{3.1}$$

BCs

$$u(0) = a,\ u(1) = b, \tag{3.2}$$

where α & β are constants, while $f(x)$ is a continuous function.

Write the *Eq. (3.1)* as

$$R(x) = \frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x} + \beta u - f(x), \tag{3.3}$$

where $R(x)$ is the residual of Eq. (3.1), and it is zero, the exact solution is known, and the boundary conditions are satisfied.

The trial series solution of *Eq. (3.1)*, $u(x)$ defined in [0, 1), satisfies the specified boundary conditions and can be extended to modified Bernoulli wavelets with unknown coefficients as follows:

$$u(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x), \tag{3.4}$$

where $c_{n,m}$'s are unknown coefficients and are to be determined.

By selecting higher-degree Bernoulli wavelet polynomials, the precision of the solution is enhanced.

Differentiate *Eq. (3.4)* w.r.t. x twice to get the values of $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ and put these values in *Eq. (3.3).*

Determining the unknown coefficients $c_{n,m}$'s, by selecting weight functions as the assumed basis elements and performing integration on boundary values along with the residual to achieve zero [8].

i.e. $\int_0^1 \psi_{1,m}(x)\ R(x)\ dx = 0,$

$m = 0, 1, 2, \ldots\ldots..$

From the above equation, derive a set of linear algebraic equations. Solving this system allows us to determine the unknown coefficients. Subsequently, by substituting these unknowns into the trial solution i.e., in *Eq. (3.4)* and obtained the numerical solution for *Eq. (3.1).*

To determine the accuracy of the WRMBW on the test problems, we use a measure of error, i.e., the maximum absolute error. The calculation of the maximum absolute error is:

$$E_{max} = \max | u(x)_e - u(x)_n |,$$

where $u(x)_e$ and $u(x)_n$ are exact and numerical solutions, respectively.

4 | Numerical Implementation

Problem (4.1): first, consider the differential equation [9],

$$\frac{\partial^2 u}{\partial x^2} + u = -x, \ 0 \le x \le 1, \tag{4.1}$$

BCs

$$u(0) = 0, \ u(1) = 0. \tag{4.2}$$

Eq. (4.1) should be implemented according to the method described in Section 3:

From *Eq. (4.1)*, the residual is given as:

$$R(x) = \frac{\partial^2 u}{\partial x^2} + u + x. \tag{4.3}$$

Then, the weight function $w(x) = x(1-x)$ should be selected for Bernoulli wavelet bases in order to meet the specified boundary conditions *Eq. (4.2),*

I. $\psi_{1,0}(x) = \psi_{1,0}(x) \times x(1-x) = x(1-x)$.

II. $\psi_{1,1}(x) = \psi_{1,1}(x) \times x(1-x) = \sqrt{3}(2x-1)x(1-x)$.

III. $\psi_{1,2}(x) = \psi_{1,2}(x) \times x(1-x) = \sqrt{5}(6x^2 - 6x + 1)x(1-x)$.

Assuming the trial solution of *Eq. (4.1)* for $k=1$ and $m=2$ is given by

$$u(x) = c_{1,0}\, \psi_{1,0}(x) + c_{1,1}\, \psi_{1,1}(x) + c_{1,2}\, \psi_{1,2}(x). \tag{4.4}$$

Then *Eq. (4.4)* becomes

$$\begin{aligned} u(x) = {} & c_{1,0}\{x(1-x)\} + c_{1,1}\{\sqrt{3}(2x-1)x(1-x)\} \\ & + c_{1,2}\{\sqrt{5}(6x^2 - 6x + 1)x(1-x)\}. \end{aligned} \tag{4.5}$$

By differentiating *Eq. (4.5)* twice with respect to variable x and substituting the values of $u, \frac{\partial^2 u}{\partial x^2}$ into *Eq. (4.3)* and derive the residual of *Eq. (4.1)*. If the weight functions are equivalent to the basis functions in the trial solution, then advance to the subsequent considerations utilizing the weighted residual method as:

$$\int_0^1 \psi_{1,j}(x)\, R(x) dx = 0, \ j = 0, 1, 2. \tag{4.6}$$

For $j = 0, 1, 2$ in *Eq. (4.6)*,

$$\text{i.e.}\quad \left.\begin{aligned} &\int_0^1 \psi_{1,0}(x)\,R(x)dx = 0, \\ &\int_0^1 \psi_{1,1}(x)\,R(x)dx = 0, \\ &\int_0^1 \psi_{1,2}(x)\,R(x)dx = 0, \end{aligned}\right\}. \qquad (4.7)$$

From *Eq. (4.7)*, a system of algebraic equations is formed that includes unknown coefficients $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$. Solving this system and obtaining values of $c_{1,0} = 0.2769$, $c_{1,1} = 0.0493$ and $c_{1,2} = -0.0018$. Substitute these values in *Eq. (4.5)* to get the numerical solution of *Eq. (4.1)*. In *Table*, a comparison of the numerical solution and the absolute errors, and *Fig. 1*, the numerical solution alongside the exact solution of equation *Eq. (4.1)* $u(x) = \frac{\sin(x)}{\sin(1)} - x$ is presented.

Table 1. Comparison of the numerical solution and the absolute errors with the exact solution of Problem (4.1).

X	Numerical Solution			Exact Solution	Absolute Error		
	FDM	Ref [9]	WRMBW		FDM	Ref [9]	WRMBW
0.1	0.018660	0.0186708	0.018628	0.0186420	1.80e-05	2.88e-5	1.40e-05
0.2	0.036132	0.0361655	0.0360910	0.0360977	3.40e-05	6.78e-5	3.20e-05
0.3	0.051243	0.0512714	0.051200	0.0511948	4.80e-05	7.66e-5	5.00e-06
0.4	0.062842	0.0628316	0.062780	0.0627829	5.90e-05	4.87e-5	3.00e-06
0.5	0.069812	0.0697452	0.069730	0.0697470	6.50e-05	1.84e-6	1.70e-05
0.6	0.071084	0.0709672	0.070980	0.0710184	6.60e-05	5.12e-5	3.80e-05
0.7	0.065646	0.0655087	0.065540	0.0655851	6.10e-05	7.64e-5	4.50e-05
0.8	0.052550	0.0524367	0.052480	0.0525025	4.80e-05	6.58e-5	2.20e-05
0.9	0.030930	0.0308742	0.030900	0.0309019	2.80e-05	2.77e-5	2.00e-06

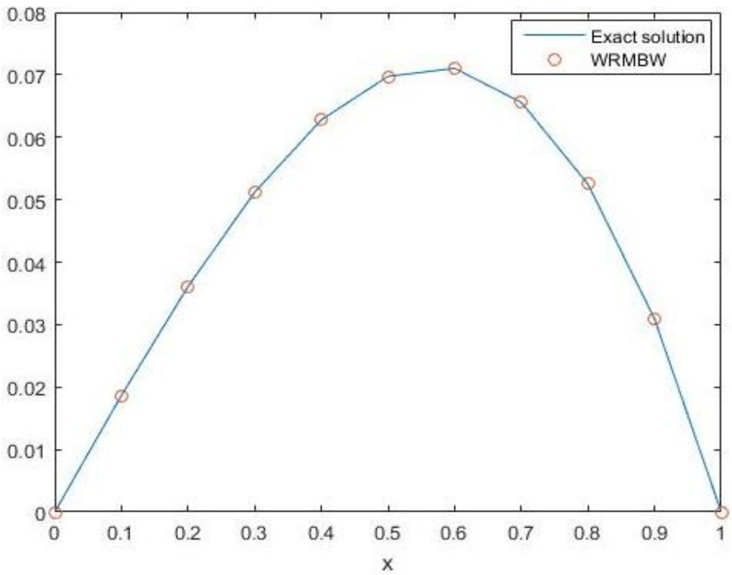

Fig. 1. Comparison of the numerical solution with the exact solution for Problem (4.1).

Problem (4.2): next, another differential equation of the form [9],

$$\frac{\partial^2 u}{\partial x^2} - \pi^2 u = -2\pi^2 \sin(\pi x),\ 0 \le x \le 1. \qquad (4.8)$$

BCs

$$u(0) = 0,\ \ u(1) = 0. \qquad (4.9)$$

As explained in Section 3 and the previous problem, finding the values of $c_{1,0}=3.7020$, $c_{1,1}=0.0$, and $c_{1,2}=-0.2634$. Substitute these values in *Eq. (4.5)*, then obtain the numerical solution. *Table 2* compares the numerical solution to the absolute errors, and *Fig. 2* represents the numerical solution to the exact solution of *Eq. (4.8)* as $u(x)=\sin(\pi x)$.

Table 2. Comparison of numerical solution and absolute error with the exact solution of Problem (4.2).

X	Numerical Solution			Exact Solution	Absolute Error		
	Ref [9]	Ref [10]	Wrmbw		Ref [9]	Ref [10]	Wrmbw
0.1	0.310207	0.308754	0.3087962	0.309016	1.19e-3	2.60e-04	2.20e-04
0.2	0.589551	0.588509	0.5885505	0.588772	7.79e-4	2.60e-04	2.20e-04
0.3	0.809478	0.809554	0.8092783	0.809016	4.62e-4	5.40e-04	2.60e-04
0.4	0.949592	0.950670	0.9506763	0.951056	1.46e-3	3.90e-04	3.80e-04
0.5	0.997656	0.999123	0.9991225	1.000000	2.34e-3	8.80e-04	8.80e-04
0.6	0.949592	0.950670	0.9506763	0.951056	1.46e-3	3.90e-04	3.80e-04
0.7	0.809478	0.809554	0.8092783	0.809016	4.62e-4	5.40e-04	2.60e-04
0.8	0.589551	0.588509	0.5875505	0.587785	1.77e-3	7.20e-04	2.30e-04
0.9	0.310207	0.308754	0.3087962	0.309016	1.02e-3	2.60e-04	2.20e-04

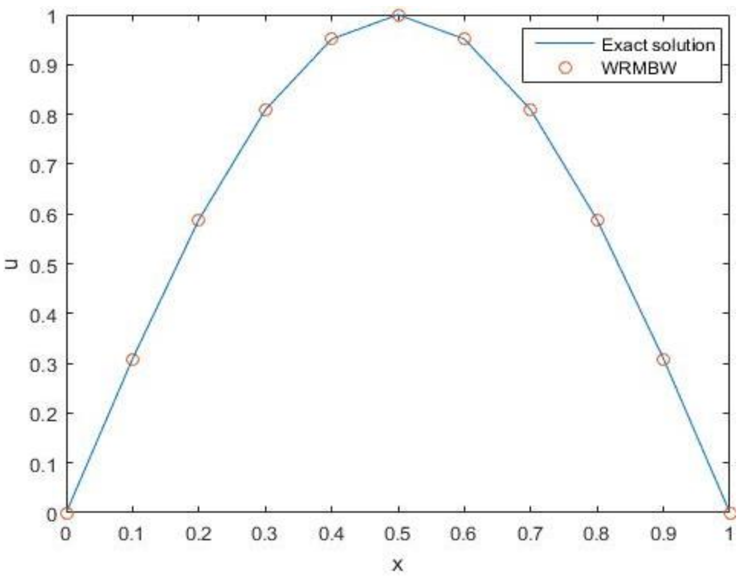

Fig. 2. Comparison between the numerical solution and the exact solution for Problem (4.2).

Problem 4.3: The differential equation is of the form [9],

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial x}=-\left(e^{x-1}+1\right), 0\le x\le 1. \tag{4.10}$$

BCs

$$u(0)=0, \quad u(1)=0. \tag{4.11}$$

As discussed in Section 3, find the values of $c_{1,0}=0.7967$, $c_{1,1}=0.1049$ and Substitute these values into *Eq. (4.5)* and then obtain the numerical solution. *Table 3* compares the numerical solution to the absolute errors and *Fig. 3* represents the numerical solution to the exact solution of *Eq. (4.10)* as $u(x)=x\left(1-e^{x-1}\right)$.

Table 3. Comparison of numerical solution and absolute error with the exact solution of Problem (4.3).

X	Numerical Solution			Exact Solution	Absolute Error		
	FDM	Ref [9]	WRMBW		FDM	Ref [9]	WRMBW
0.1	0.061948	0.059251	0.059427	0.059343	2.61e-03	9.20e-5	8.40e-05
0.2	0.115151	0.109902	0.110154	0.110134	5.02e-03	3.32e-4	2.00e-05
0.3	0.158162	0.150735	0.150983	0.151024	7.14e-03	2.89e-4	4.10e-05
0.4	0.189323	0.180249	0.180433	0.180475	8.85e-03	2.26e-4	4.20e-05
0.5	0.206737	0.196660	0.196743	0.196735	1.00e-02	7.50e-5	8.00e-06
0.6	0.208235	0.197904	0.197875	0.197808	1.04e-02	9.60e-5	6.70e-05
0.7	0.191342	0.181631	0.181507	0.181427	9.92e-03	2.04e-4	8.00e-05
0.8	0.153228	0.145212	0.145039	0.145015	8.21e-03	7.00e-4	2.40e-05
0.9	0.090672	0.085733	0.085680	0.085646	5.03e-03	4.18e-5	3.40e-05

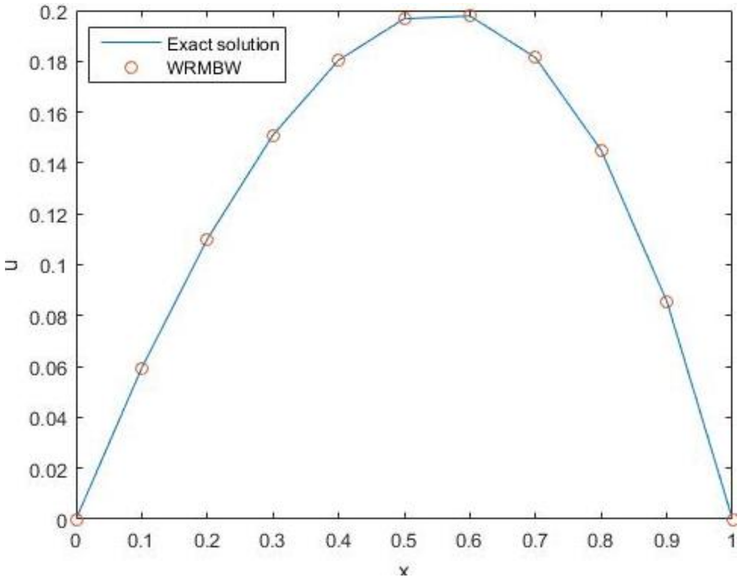

Fig. 3. Comparison between the numerical solution and the exact solution for Problem (4.3).

Problem (4.4): Finally, the non-linear differential equation [11].

$$\frac{\partial^2 u}{\partial x^2} - u^2 = 2\pi^2 \cos(2\pi x) - \sin^4(2\pi x), \; 0 \le x \le 1. \quad \textbf{(4.12)}$$

BCs

$$u(0) = 0, \quad u(1) = 0. \quad \textbf{(4.13)}$$

The exact solution of *Eq. (4.11)* $u(x) = \sin^2(\pi x)$ is shown in *Table 4* and *Fig. 4* together with the numerical solution, which was derived as described in Section 3.

X	Wrmew	Exact Solution	Absolute Error

0.1	0.097457	0.0954920	1.97E-03
0.2	0.351009	0.3454920	5.52E-03
0.3	0.657342	0.6545082	2.83E-03
0.4	0.906851	0.9045082	2.34E-03
0.5	0.997985	1	2.01E-03
0.6	0.910379	0.9045082	5.87E-03
0.7	0.658956	0.6545082	4.45E-03
0.8	0.348898	0.3454920	3.41E-03
0.9	0.097684	0.0954920	2.19E-03

Table 4. Comparison of WRMEW and absolute error with the exact solution for test Problem (4.4).

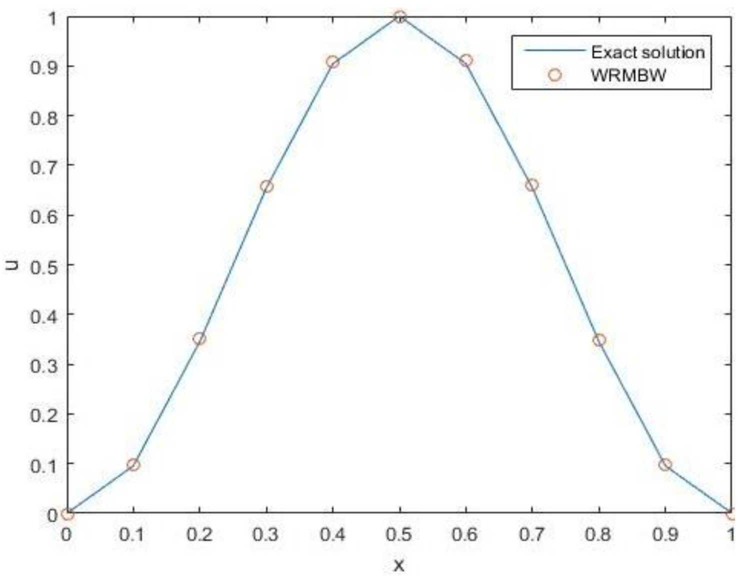

Fig. 4. Comparison of WRMEW and the exact solution for test Problem (4.4).

5 | Conclusion

This work presents the WRMBW for the numerical solution of one-dimensional differential equations. According to the information shown in the tables and figures, the numerical solutions obtained using the suggested approach outperform those obtained using the current approaches ((FDM) [9], [10]) and are closer to the exact solution. Additionally, compared to current methods, the absolute error produced by this methodology is significantly lower ((FDM), [9], [10]). Thus, the Bernoulli wavelet-based weighted residual approach has proven to be remarkably successful in solving one-dimensional differential equations.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability

All data are included in the text.

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